**Quantum Mean Field Theory**

Now we’ll proceed to a more advanced model.

**Classical Heisenberg model of vector interacting spins**

We want to study the classical *Heisenberg* model. Previously, we were studying the Ising model, which simplified the interactions between spins so that only the SZ’s were coupled. But now we’ll let the coupling take place between any two orientations of spins. So our H is:



But instead of **S** being an operator, we’ll make **S** is a *classical* spin vector of unit magnitude, which can point in any direction. We’ll presume our spins can rotate in n spatial dimensions, but exist on a lattice of d dimensions (for instance, we could have a set of n = 3 component vector spins all lined up on a straight line, i.e., a d = 1 dimensional lattice). So this is a semi-classical approximation to the quantum Heisenberg model. Our Z would be:



where we integrate each Si over its d.o.f, i.e. the solid angle differential dn-1Ω, which is dφ in 2D, dφd(cosθ) in 3D, dφd(cosθ1)dφ(dcosθ2) in 4D, etc. (can see the Stat Mech folder / Classical Paramagnet file on why d.o.f. is, e.g., dΩ = dφd(cosθ) instead of dφdθ. And of course we’ve subsumed the β appearing in Z into the coupling constants, J, and h:



**Writing as a functional integral**

Now let’s see if we can cast this as a functional integral. I’m assuming the Hubbard-Stratonovich identity will straight-forwardly generalize to n components for us at the moment):



If so then, we can write, ignoring the prefactor, which should make negligible contribution to the free energy in the large N limit,



Let’s define the spin sum/integral,



where **S** is the thing being integrated over the unit n-sphere. This can be done in certain circumstances … n = 3 is easy. n ≠ 3 isn’t so easy. For instance,



But either way, then we can write,



So then we can write,



So now we ‘have’ the functional integral representation.

**Going to continuum limit and keeping only long wavelength (small k) terms**

Now we’ll go to the continuum limit. We established already in the Ising HS file, that for n.n. interactions,



and I think we may likewise surmise (using density n0 = 1/ad),



where,



We can integrate by parts on this term and get:



So hopefully,



Might note that if K = 0, i.e., no interaction, then what we get compares well with the result for a classical spin in a magnetic field (see Stat Mech/Classical Paramagnet, and remember that we absorbed the magnetic moment and field into j), in 3D,



**Small φ expansion near the critical point**

Now let’s make an expansion near the critical point. Since φ ~ m, this means small φ, and we’ll also presume small h ~ small j. Now let’s Taylor expand the In(x) function. And need to go out to 4th order in **x**. Well, see the Appendix. We have:



where Ωn = 2(√π)n/Γ(n/2) is the solid angle of an n-sphere. And so,



So we can say, ignoring the constant,



Terms neglected are irrelevant close to the critical point, as per RG analysis. Okay then altogether,



Looking at the prefactor of the φ2 term, we can see that the critical temperature is where Kz = n. So let’s just keep terms to first non-vanishing order about Kz = n. I think I’ll do this by factoring out Kz/n, and then taking limit Kz → n. So then,



Now we’ll change to unitless variables: r → ar, and rename the resulting field variables φ(ar) → φ(r), and j(ar) → j(r). Then we’ll have:



We’d recognize the (mean field) critical temperature as where r = 0 → Kz = n → βJz = n → Tc = Jz/n. In terms of Tc, we can write r as:



(if kB is set to 1, and keeping just leading term in T - Tc) Well whatever. So then defining,



we’ll have:



In what follows we won’t presume any particular dimension, or number of components. Recall from a previous file, that we can get the magnetization via:



So,



The generalized susceptibility/Green’s function would be given by:



where φ0(a) = ma(x) = <φa(x)> and is presumed to not depend on x. In the disordered phase φ0(a) is zero, in which case this would just be β<φa(x)φb(x´)>. In any event. This guy is just β × the exact two-point GF.



And recall the critical exponents we can get from χ/G,



And also recall (Thermodynamics folder/Critical Exponents) that from the Fourier transform of χ(r-r´), we have the uniform susceptibility, and another critical exponent,



as well as the critical temperature I guess. And of course with these critical exponents in hand, we can obtain all others using the Widom scaling hypothesis stuff (see Thermodynamics/Critical Exponents).

**Exact MFT limit**

So we can study the saddle point solution to this action. This would be the mean field result. Instead, let’s do this. We’ll go to the large n limit (n is number of vector components, recall), and show that mean field theory is exact in this limit. But as we’ll see in the next file, we have to scale the interaction u → u/n to keep our action from being pathological. So then we have:



And go back to the partition function,



Now let’s decouple the φ4 term using the HS transformation again (see Path Integrals file or just note ∫exp(-ax2-bx-c) ~ exp(b2/4a-c)). We are ignoring the determinant prefactor in front of the ∫D[σ(x)]exp{…} term, as it doesn’t really matter – it’s independent of φ.



So we have:



Then, noting D**φ** = D[φ1(x)]D[φ2(x)]…D[φn(x)] we can write this as:



So now we can see that in the large n limit, the saddle point approximation is exact. Well let’s do the Dφ1 integral (see Path Integrals file for more careful derivation of such things).



So at present we have:



Now we can do a saddle point evaluation of this integral too, which should be exact in the large n limit, thanks to the n in front of the σ2 term. I’m going to leave off the j’s though. And I’ll just evaluate,



Taking a functional derivative,



Hmmm….let me try this another way. Let’s go back to:



And we’ll just do a saddle point analysis here.



and,



So our two equations are:



Filling the latter into the former, we have:



If we take the functional derivative w/r to j1(x´), then we’d have:



[δφ/δh = χ = βG, and δφ/δj = G] Now φ1(x) is just m1(x), in our saddle point analysis. So, we have:



which is the same equation, basically, that we had for the GF in the inhomogeneous mean field Ising model. So we can expect the same critical behavior.

**Appendix**

So we want the expansion of,



to fourth order in **a**. And we’ll note that only the even order terms will contribute. It looks like one has to explicitly do this? So apparently:



Therefore,



where,



Now since,



we can write,



at which point, we’ll use the identity involving the β function:



to write,



And now we can do some cancellation, and use Γ(1/2) = √π.



Now using the recursion relation Γ(z+1) = zΓ(z), we have,



We’ll recognize the solid angle of an n-sphere Ωn = In(0) = 2(√π)n/Γ(n/2). And so we can write,



And so our expansion is,



For n = 3, we find,



which matches the exact result,



So that’s reassuring.